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CALCULATION OF THE LIMIT-EQUILIBRIUM OF RETAINED VISCOPLASTIC OIL EXTRACTED FROM A NONUNIFORM STRATIFIED LAYER BY WATER *

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The theory of calculation of limit-equilibrium retained viscoplastic oil /1—4/ is extended to the case of nonuniformly stratified layers with free flow between the strata. The ensuing problem is reduced to a system of nonlinear integral equations. Examples of derivation of approximate analytic and numerical solutions are presented.

1. Let us assume that the oil initially contained in a thin stratified layer was extracted from the latter for a fairly long time, remaining only in those parts of the layer where the pressure gradient is below the limit pressure gradient for oil (G_i in the i-th intercalation). In the region occupied by water its motion conforms to the Darcy law and the pressure distribution satisfies the equation

$$\frac{\partial}{\partial x}\left(k\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial p}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial p}{\partial z}\right)=0 \tag{1.1}$$

where k is the effective penetrability of water after the extraction of oil assumed to be a piecewise constant function of coordinate z ($k=k_i$ for $z_i<z<z_{i+1}$). The intercalations are numbered in the order of decreasing permeability ($k_1>k_2>\ldots>k_n$) so that $G_1<G_2<\ldots<G_n$. We further assume that the boundaries of retained oil in each intercalation have the form of cylindrical surfaces with their generatrices parallel to the z-axis (this assumption may be removed, but then the problem is to be formulated somewhat differently).

We denote by D_j the projection of the retained oil in the j-th intercalation on the x, y-plane. The projections of retained oil occupy in that plane regions that increase with decreasing permeability of intercalations $D_j\subset D_{j+1}$. The motion of water in the region free of retained oil is the same as in a nonuniformly stratified layer of piecewise constant thickness. Assuming a perfect intercommunication between intercalations and averaging the motion over the layer thickness, we obtain (cf. /5/)

$$\operatorname{grad} p=-\frac{\mu(h_1+h_2+\ldots+h_n)}{k_1h_1+k_2h_2+\ldots+k_jh_j}\,\mathbf{w} \tag{1.2}$$

$$\operatorname{div}\mathbf{w}=0\quad (M(x,y)\in D_{j+1}/D_j)$$

where $\mathbf{w}$ is the mean filtration rate over the layer thickness, p is the pressure normalized with respect to some pressure head, and the vector operations are considered in the x, y-plane. The number j up to which summation is carried out in (1.2) is determined by the part of intercalations (with $j+1$ to n) occupied by retained oil at point (x, y) of the layer. Thus for a given disposition of retained oil zones the averaged motion of water is defined by equations of a plane filtration stream in a piecewise inhomogeneous medium, with projections of retained oil boundaries on the x, y-plane (lines Γ_j) - to be determined in the course of solution of the problem - play the part of boundaries between regions of different effective permeability.

The additional conditions required for determining the Γ_j boundaries follow from the physically obvious limit equilibrium condition: the retained oil equilibrium in the j-th intercalation is possible provided the pressure gradient in it does not exceed in absolute value G_j, with the equilibrium taken as limiting in the sense that at the retained oil boundary the pressure gradient is at its limit value

$$|\operatorname{grad} p(x,y)|=G_j,\quad (x,y)\in\Gamma_j \tag{1.3}$$

This condition means that with the least increase of the pressure gradient (rate of extraction) the pressure gradient exceeds at least in a part of the retained oil its limit value, and the oil begins to move. For the Γ_1 boundary of region in which water does not move, condition (1.3) is the same as used earlier in the theory of retained oil in homogeneous layers.

The meaning of condition (1.3) at the remaining boundaries of retained oil is explained as follows. In conformity with formulas (1.2) the usual conditions of continuity of pressure and flow

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$$p^+ = p^-, \quad \left(k\frac{\partial p}{\partial n}\right)^+ = \left(k\frac{\partial p}{\partial n}\right)^- \tag{1.4}$$

are satisfied at these boundaries. Here, the superscripts plus and minus denote limit values of pressure, its normal derivative, and effective permeability close to the boundary approached from inside and outside, respectively.

Note that in the used here scheme of flow $k^+ < k^-$, hence $\partial p^+ / \partial n > \partial p^- / \partial n$. Differentiation of the first of equalities (1.4) along the tangent to the contour yields $\partial p^+ / \partial s = \partial p^- / \partial s$. We thus have $|\mathrm{grad}\, p^+| > |\mathrm{grad}\, p^-|$. Taking this inequality into account, we come to the fundamental conclusion: the use for the determination of boundaries of retained oil zones Γ_j of the limit equilibrium condition (1.3) it is necessary to substitute into it the absolute values of the pressure gradient inside the region D_j occupied by the retained oil. The limit pressure gradient calculated close to the retained oil boundary on the outside of the latter is lower than the threshold value of G_j.

This conclusion may appear paradoxical, since in the j-the intercalation the retained oil is washed on the outside by a stream in which the pressure gradient at the retained oil boundary is below its limit value for that intercalation, and it would seem possible to increase the retained oil zone size without upsetting its limit equilibrium. However, when the retained oil zone is widened, the pressure gradient exceeds the limit value not in the external region of the retained oil but in the region motion of water below (or above) the latter. Because of the assumption about the hydrostatic pressure distribution over the layer thickness, the pressure gradient inside the retained oil exceeds the limit, and this contradicts the initial assumptions.

The additional conditions for the determination of retained oil boundaries are, consequently, of the form

$$|\mathrm{grad}\, p^-| = G_1, \quad x, y \in \Gamma_1$$
$$|\mathrm{grad}\, p^+| = G_j, \quad x, y \in \Gamma_j, \quad j > 1 \tag{1.5}$$

Below, we consider the case of a two-strata layer, assuming that the limit gradient for the most permeable intercalation can be neglected, i.e. $G_1 = 0$ and $G_2 = G$.

2. As in other plane problems of nonlinear filtration (cf. /4,5/) the problem formulated in Sect.1 admits a class of exact elementary solutions with the stagnation point at the coordinate origin

$$\begin{aligned} &p = Ar^s \cos(s\theta), \quad r < R, \quad 0 \leqslant \theta \leqslant \pi/s \\ &p = (Br^s + Cr^{-s}) \cos(s\theta), \quad R < r < \infty \\ &R = \left(\frac{G}{As}\right)^{1/(s-1)}, \quad B = \frac{G}{s} R^{1-s} \frac{1}{1-\lambda} \\ &C = -\frac{G}{s} R^{s+1} \frac{\lambda}{1-\lambda}, \quad \lambda = \frac{k^+ - k^-}{k^+ + k^-} \end{aligned} \tag{2.1}$$

and at infinity

$$\begin{aligned} &p = (Ar^s + Br^{-s}) \cos(s\theta), \quad r < R \\ &p = Cr^{-s} \cos(s\theta), \quad r > R, \quad 0 < \theta < \pi/s \\ &R = \left(\frac{G}{Cs}\right)^{1/(s+1)}, \quad A = -\frac{G}{s} R^{1-s} \frac{\lambda}{1-\lambda} \\ &B = \frac{G}{s} R^{1+s} \frac{1}{1-\lambda}, \quad s > \frac{1}{2} \end{aligned} \tag{2.2}$$

where r and θ are polar coordinates. For $s > 1$ solution (2.1) corresponds to a flow inside an angle of opening π/s and a singularity at infinity. The retained oil occupies a sector of radius R adjacent to the angle apex. For $1/2 < s < 1$ we have a flow inside an angle of opening $\theta_0 = \pi/s > \pi$ (external flow over a wedge) with the retained oil occupying the exterior of sector of radius R. Solution (2.2) relates to a flow inside and angle of opening π/s with a singularity of the type of generalized dipole at the angle apex, with the retained oil occupying the exterior of the sector of radius R.

A trivial plane-parallel flow corresponding to a source at the coordinate origin, also exists.

3. Using elementary solutions and the technique of merging asymptotic expansions /6/, it is possible to derive for a number of problems approximate solutions that correspond to high and low stream intensities. Let us show this on the example of flow produced by n batteries of equal capacity apertures. In the absence of a limit gradient $(G = 0)$ the respective complex potential is of the form

$$p + i\psi = W(\zeta)\frac{\mu}{kh} = \frac{q\mu}{2\pi kh}\ln(\zeta^n - a^n) \tag{3.1}$$

where q is the source intensity, μ is the fluid viscosity, and k is the mean permeability of the layer. Since the flow (3.1) has stagnation points at $\zeta = 0$ and infinity, hence for small but finite G the appearance of two retained oil zones: a small internal one in the coordinate origin neighborhood, and a large one in the neighborhood of an infinitely distant point.

Let us examine the internal retained oil neighborhood. The external solution of the problem that corresponds to $G = 0$ is of the form (3.1) and its internal asymptotics is of the form

$$W(\zeta) = C - \frac{q}{2\pi}\frac{\zeta^n}{a^n} + \ldots = C - \frac{q}{2\pi}\frac{r^n}{a^n}e^{in\theta} \quad (\zeta \to 0) \tag{3.2}$$

The internal solution corresponds to the retained oil in the stagnation point neighborhood at the coordinate origin and is provided by formulas (2.1). Its external asymptotics

$$p = Br^s \cos(s\theta) + \ldots \quad (r \to \infty) \tag{3.3}$$

The condition of matching asymptotics (3.2) and (3.3) implies that

$$s = n, \quad B = \frac{\mu q}{2k\pi h a^n}$$

with all parameters of the internal solution determined by formulas (3.1). For the retained oil zone radius we have

$$\frac{R}{a} = \left[\frac{2\pi k^- a G}{n\mu q}\frac{1}{1-\lambda}\right]^{1/(n-1)}$$

Since at infinity the initial flow asymptotically coincides with the flow from the source of intensity nq, for the inner radius of the external retained oil zone we obtain

$$R_1 = \frac{hq\mu}{2\pi k^- G}\frac{1-\lambda}{1+\lambda}$$

4. The method of matching asymptotic expansions using elementary solutions enables us to obtain only the principal terms of asymptotics at low and high stream intensities. In a more general case the problem can be reduced to a system of nonlinear integral equations that can be solved by asymptotic or numerical methods. We shall show this on the example of the flow produced in a two-strata layer by the source-sink pair of the same capacity Q, and located at points $(a, 0)$ and $(-a, 0)$. With fairly large Q a single retained oil zone is generated; it occupies the exterior of the oval contour Γ inside which lie the source and sink.

The sought solution may be represented in the form of superposition of the potential $\varphi(x, y)$ of the source-sink pair on the potential of a simple layer of density $\nu(\sigma)$ concentrated on the contour Γ. The idea of using integro-differential equations for determining the form of an unknown contour was given to the authors by the works of V. L. Danilov and his disciples /see, e.g., /1/). Then

$$p(x, y) = \varphi(x, y) + \int_\Gamma \nu(\sigma)\ln\frac{1}{R}\,d\sigma$$

$$\varphi(x, y) = -\frac{Q\mu}{4\pi k^-}\ln\frac{(x+a)^2+y^2}{(x-a)^2+y^2}$$

$$R^2 = (x-\xi)^2 + (y-\eta)^2, \quad (\xi, \eta) \in \Gamma$$

In this case $p(x, y)$ continuously varies at transition through the contour Γ, and has the specified singularities at apertures. The conditions of equality of streams and of limit equilibrium yield for the determination of density $\nu(x, y)$ and form of contour Γ the following system of integro-differential equations:

$$\frac{\pi}{\lambda}\nu + \int_\Gamma \nu(\sigma)\frac{\partial}{\partial n}\ln\frac{1}{R}\,d\sigma + \frac{\partial\varphi}{\partial n} = 0 \tag{4.1}$$

$$\frac{\partial\varphi}{\partial s} + \int_\Gamma \nu(\sigma)\frac{\partial}{\partial s}\ln\frac{1}{R}\,d\sigma = -\left[G^2 - \frac{\pi^2(1-\lambda)^2}{\lambda^2}\nu^2\right]^{1/2}$$

In deriving system (4.1) allowance was made for the properties of derivatives of the simple layer potential, and the normal was assumed to be internal.

We pass in system (4.1) to polar coordinates, r, θ. Assuming the contour Γ to be defined by the equation $r = g(\theta)$, we obtain

$$\frac{\pi}{\lambda}\nu(\theta) + \int_0^{2\pi}\nu(\alpha)\frac{L(\alpha)}{L(\theta)}K_0[\alpha, \theta, g(\alpha), g(\theta)]\,d\alpha = \tag{4.2}$$

$$-\frac{ac}{L(\theta)\,l^2[\theta, g\theta]}\left[g(\theta)(a^2 - g^2(\theta))\cos\theta + g'(\theta)(a^2 + g^2(\theta))\sin\theta\right]$$

$$\int_0^{2\pi} \nu(\alpha)\frac{L(\alpha)}{L(\theta)} K_1[\alpha, \theta, g(\alpha), g(\theta)]\, d\alpha + \left[G^2 - \frac{\pi^2(1-\lambda)^2}{\lambda^2}\nu^2(\theta)\right]^{1/2} =$$

$$-\frac{ac}{L(\theta)\, l^2[\theta, g(\theta)]}[g'(\theta)(a^2 - g^2(\theta))\cos\theta - (a^2 + g^2(\theta))\, g(\theta)\sin\theta]$$

where

$$K_0[\alpha, \theta, g(\alpha), g(\theta)] = \frac{g^2(\theta) - g(\theta) g(\alpha)\cos(\theta-\alpha) - g'(\theta) g(\alpha)\sin(\theta-\alpha)}{\varkappa^2(\alpha, \theta)}$$

$$K_1[\alpha, \theta, g(\alpha), g(\theta)] = \frac{g(\theta) g'(\theta) - g(\alpha) g'(\theta)\cos(\theta-\alpha) + g(\alpha) g(\theta)\sin(\theta-\alpha)}{\varkappa^2(\alpha, \theta)}$$

$$\varkappa^2(\alpha, \theta) = g^2(\theta) - 2g(\theta) g(\alpha)\cos(\theta-\alpha) + g^2(\alpha),$$

$$l^2[\theta, g(\theta)] = a^4 - 2a^2 g^2(\theta)\cos 2\theta + g^4(\theta)$$

$$L(\theta) = [g^2(\theta) + g'^2(\theta)]^{1/2}, \quad c = Q\mu / (\pi k^-)$$

Setting in system (4.2)

$$\nu(\theta) = Gm(\theta), \quad g(\theta) = \sqrt{\frac{Qa}{\pi G}}\, q(\theta), \quad \varepsilon = \frac{\pi a k^- G}{Q\mu}$$

we derive from it a system of the same form in which m, q, and ε are substituted for ν, g, and a^2, respectively, and G and ac made equal unity.

We seek a solution of this system taking ε as the small parameter, i.e. considering the flow to be of fairly high intensity. Setting $\varepsilon = 0$ we obtain the following solution:

$$m(\theta) = \frac{\lambda}{\pi(1-\lambda)}\cos\theta, \quad q(\theta) = q_0 = \sqrt{1-\lambda}$$

which corresponds to a solution which is asymptotic in the meaning of Sect.2. The retained oil zone has then the form of a circle of radius $R = [Qa(1-\lambda)\mu / (\pi G k^-)]^{1/2}$. For small ε we set

$$m(\theta) = \frac{\lambda}{\pi(1-\lambda)}\cos\theta + \varepsilon m_1(\theta), \quad q(\theta) = q_0 + \varepsilon q_1(\theta)$$

and obtain

$$K_0 = \frac{1}{2} + \frac{\varepsilon}{2q_0}\frac{[q_1(\theta) - q_1(\alpha)]\cos(\theta-\alpha) - q_1'(\theta)\sin(\theta-\alpha)}{1-\cos(\theta-\alpha)}$$

$$K_1 = \frac{1}{2}\left[\operatorname{ctg}\frac{\theta-\alpha}{2} + \varepsilon\frac{q_1(\theta)}{q_0}\right], \quad \frac{L(\alpha)}{L(\theta)} = 1 + \varepsilon\frac{q_1(\alpha) - q_1(\theta)}{q_0}$$

From system (4.2) expressed in dimensionless variables we obtain the following linear system of singular integro-differential equations:

$$\frac{\pi}{\lambda} m_1(\theta) + \frac{1}{2}\int_0^{2\pi} m_1(\alpha)\, d\alpha + \frac{\lambda}{2\pi(1-\lambda)}\int_0^{2\pi}\cos\alpha \times$$
$$\frac{[p(\theta) - p(\alpha)]\cos(\theta-\alpha) - p'(\theta)\sin(\theta-\alpha)}{1-\cos(\theta-\alpha)}\, d\alpha +$$
$$\frac{\cos\theta}{1-\lambda}\left[\frac{1}{1-\lambda} - \frac{2}{1-\lambda}\cos 2\theta + 2p(\theta)\right] + \frac{p'(\theta)\sin\theta}{1-\lambda} = 0$$

$$\frac{1}{2}\int_0^{2\pi} m_1(\alpha)\operatorname{ctg}\frac{\theta-\alpha}{2}\, d\alpha - \frac{\lambda}{2\pi(1-\lambda)}\int_0^{2\pi}\cos\alpha\,[p(\theta) - p(\alpha)] \times$$
$$\operatorname{ctg}\frac{\theta-\alpha}{2}\, d\alpha - \frac{\sin\theta}{1-\lambda}\left[\frac{1+2\cos 2\theta}{1-\lambda} - 2p(\theta)\right] -$$
$$\frac{p'(\theta)\cos\theta}{1-\lambda} = \frac{\pi(1-\lambda)}{\lambda} m_1(\theta)\operatorname{ctg}\theta$$

$$p(\theta) = q_1(\theta) / q_0$$

whose solution accurate to within terms with second powers of ε is of the form

$$q(\theta) = \sqrt{1-\lambda} + \varepsilon\frac{2\cos 2\theta - \lambda}{4\sqrt{1-\lambda}}$$

$$m(\theta) = \frac{\lambda}{\pi(1-\lambda)}\cos\theta$$

5. The same treatment can be applied with minor modifications to other more complex problems of determination of the retained oil zone dimensions. In such problems it is essential to know beforehand the number and general disposition of retained oil zones. Thus in the case of flow generated by two sources of the same intensity Q with a fairly small parameter $\varepsilon = (\pi a k^- G / Q\mu)^2$ two retained oil zones are formed: the inner zone bounded by contour Γ_0 and the outer one bounded from inside by contour Γ_1 (Fig.1).

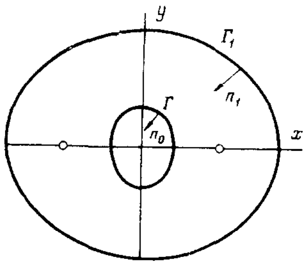

Fig. 1

Introducing the potentials of the simple layer of densities ν_0 and ν_1, respectively, we can reduce the problem to the solution of the system of equations in dimensionless form

$$\frac{\pi}{\lambda} m(\theta) + \int_0^{2\pi} m(\alpha)\frac{L(\alpha)}{L(\theta)} K_0[\alpha, \theta, q(\alpha), q(\theta)]\,d\alpha - \tag{5.1}$$

$$\frac{1}{\varepsilon}\int_0^{2\pi} M(\alpha)\frac{L_1(\alpha)}{L(\theta)} K_0[\alpha, \theta, f(\alpha), \varepsilon q(\theta)]\,d\alpha +$$

$$\frac{a^4[\varepsilon q^4(\theta) - q^2(\theta)\cos 2\theta - q(\theta)q'(\theta)\sin 2\theta]}{L(\theta)\, l^2[\theta, a\sqrt{\varepsilon} q(\theta)]} = 0$$

$$\int_0^{2\pi} m(\alpha)\frac{L(\alpha)}{L(\theta)} K_1[\alpha, \theta, q(\alpha), q(\theta)]\,d\alpha +$$

$$\frac{1}{\varepsilon}\int_0^{2\pi} M(\alpha)\frac{L_1(\alpha)}{L(\theta)} K_1[\alpha, \theta, f(\alpha), \varepsilon q(\theta)]\,d\alpha \pm$$

$$\left[1 - \frac{\pi^2(1-\lambda)^2}{\lambda^2} m^2(\theta)\right]^{1/2} +$$

$$\frac{a^4[\varepsilon q^3(\theta)q'(\theta) - q(\theta)q'(\theta)\cos 2\theta + q^2(\theta)\sin 2\theta]}{L(\theta)\, l^2[\theta, a\sqrt{\varepsilon} q(\theta)]} = 0$$

$$\frac{\pi}{\lambda} M(\theta) + \int_0^{2\pi} M(\alpha)\frac{L_1(\alpha)}{L_1(\theta)} K_0[\alpha, \theta, f(\alpha), f(\theta)]\,d\alpha +$$

$$\varepsilon\int_0^{2\pi} m(\alpha)\frac{L(\alpha)}{L_1(\theta)} K_0[\alpha, \theta, \varepsilon q(\alpha), f(\theta)]\,d\alpha +$$

$$\frac{a^4[f^4(\theta) - \varepsilon^2 f^2(\theta)\cos 2\theta - \varepsilon f(\theta)f'(\theta)\sin 2\theta]}{\varepsilon^2 L_1(\theta)\, l^2[\theta, af(\theta)/\sqrt{\varepsilon}]} = 0$$

$$\int_0^{2\pi} M(\alpha)\frac{L_1(\alpha)}{L_1(\theta)} K_1[\alpha, \theta, f(\alpha), f(\theta)]\,d\alpha +$$

$$\varepsilon\int_0^{2\pi} m(\alpha)\frac{L(\alpha)}{L_1(\theta)} K_1[\alpha, \theta, \varepsilon q(\alpha), f(\theta)]\,d\alpha \pm$$

$$\left[1 - \frac{\pi^2(1-\lambda)^2}{\lambda^2} M^2(\theta)\right]^{1/2} +$$

$$\frac{a^4[f^3(\theta)f'(\theta) - \varepsilon f(\theta)f'(\theta)\cos 2\theta + \varepsilon f^2(\theta)\sin 2\theta]}{\varepsilon^2 L(\theta)\, l^2[\theta, af(\theta)/\sqrt{\varepsilon}]} = 0$$

$$L_1(\theta) = [f^2(\theta) + f'^2(\theta)]^{1/2}, \quad \varepsilon = \left(\frac{\pi k^- aG}{\mu Q}\right)^2$$

$$\Bigl(g(\theta) = a\sqrt{\varepsilon} q(\theta), \quad F(\theta) = \frac{a}{\sqrt{\varepsilon}} f(\theta)$$

$$\nu_0 = \frac{Q\mu}{a\pi k^-}\sqrt{\varepsilon} m(\theta), \quad \nu_1 = \frac{Q\mu}{a\pi k^-}\sqrt{\varepsilon} M(\theta)\Bigr)$$

The polar equations of the inner and outer contours and their respective densities of the simple layer potentials appear in these equations in parentheses.

It will be readily seen that the input system decomposes with an accuracy to therms of order ε^2 in two independent systems for the determination of $q(\theta)$ and $m(\theta)$ and, respectively, $f(\theta)$ and $M(\theta)$. Carrying out the expansion in the small parameter as in Sect.4, we obtain the approximate solution of these systems. In dimensional variables they are of the form

$$g(\theta) = \frac{\pi a^2 k^- G}{Q\mu(1-\lambda)}\left[1 - \frac{\varepsilon}{(1-\lambda)^2}\cos 2\theta\right]$$

$$\nu(\theta) = -\frac{\lambda G}{\pi(1-\lambda)}\left[\cos 2\theta + \frac{\varepsilon}{(1-\lambda)^2}\sin^2 2\theta\right]$$

$$F(\theta) = \frac{1-\lambda}{1+\lambda}\frac{Q\mu}{\pi k^- G}, \quad \nu_1(\theta) = -\frac{\lambda G}{\pi(1-\lambda)}$$

6. The problems (4.2) and (5.1) were also solved numerically. The effect of the external contour was not taken into account in the solution of (5.1). The input system of equations of each problem for the unknown contour $g(\theta)$ and the potential densities $\nu(\theta)$ along it were defined as follows:

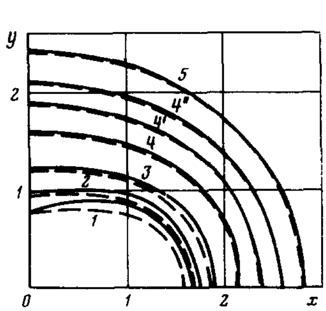

Fig.2

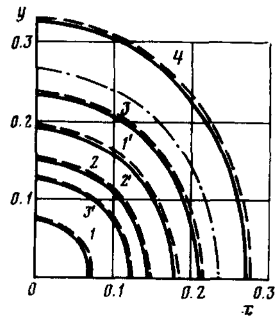

Fig.3

$$\nu(\theta) = \frac{\lambda}{\pi}\left(\frac{\partial\varphi}{\partial n} + \int_0^{2\pi} \nu(\alpha) K_0(\alpha, \theta)\, d\alpha\right), \quad \left(\frac{\partial p}{\partial n}\right)^2 + \left(\frac{\partial p}{\partial s}\right)^2 = G^2 \tag{6.1}$$

$$\frac{\partial p}{\partial n} = \frac{\pi(1-\lambda)}{\lambda}\frac{\nu(\theta)}{L(\theta)}, \quad \frac{\partial p}{\partial s} = \frac{\partial\varphi}{\partial s} + \int_0^{2\pi} \nu(\alpha) K_1(\theta, \alpha)\, d\alpha$$

where $\varphi(r, \theta)$ is the potential of the pair corresponding to each problem.

The method of successive approximations was used for solving the first equation of system (6.1) which for the specified contour is a linear integral Fredhol equation of the second kind in $\nu(\theta)$. The sought contour $g(\theta)$ was determined using the second equation of the system, which together with the first was solved by the simplest iteration method of the shooting type. The approximate analytic solution, as well as the exact solution for $\lambda = 0$ were used as the input approximation. In each example computations were carried out until the result matched the contours obtained in the two initial approximations with the specified accuracy. For reducing the volume of computations the respective symmetry was used in both problems.

Computations were carried out for various values of parameter ε and $\delta = k^- / k^+$ on which solutions of both problems depend. The solutions for the system source—sink are shown in Fig. 2 and for the system source—source in Fig.3 in dimensionless variables. The boundaries of limit—equilibrium retained oil zones appearing there were obtained by several methods. The solid lines relate to the approximate analytic solution and the dash lines to the numerically computed. Curves *1–5* in Fig.2 correspond to $\varepsilon = 1, 0.8, 0.6, 0.4, 0.2$ and $\delta = 2$, while curves *4'* and *4"* relate to $\varepsilon = 0.4$ and $\delta = 5, 100$. Curves *1–4* in Fig.3 correspond to $\varepsilon = 0.1, 0.2, 0.3, 0.4$ with $\delta = 2$, as previously, while curves *1'*, *2'*, and *3'* are for $\varepsilon = 0.25$ and $\delta = 2, 5, 100$; the dash-dot curve represents here the exact solution of the problem ($\lambda = 0$) with $\varepsilon = 0.25$.

These results show a good agreement of the numerical and the approximate analytic solutions for both problems, with the discrepancy decreasing with decreasing ε and the ratio δ maintained constant, while in the case of fixed parameter ε it decreases with decreasing δ. The magnitude of the retained oil zone determined by the approximate analytic solution is smaller than the numerically computed. The over-all magnitude of this discrepancy is shown in Fig.4, where the relative size of the surface area Δ between the two contours is represented in terms of ε, obtained by various methods. Curve *1* corresponds here to the problem with the pair source—sink, and curve *2* to that of the pair source—source. It will be seen that for $\varepsilon < 0.2$ the quantity Δ is virtually zero, becoming appreciable ($> 2\%$) only for $\varepsilon > 0.4$.

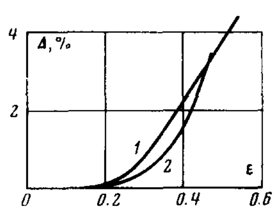

Fig. 4

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